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$$(x^2+a^2)\tan^2(90^\circ-\theta)=(h-z)^2.$$

If this be a rectangular hyperbola,

$$\tan^2(90^\circ-\theta)=1 \qquad \tan(90^\circ-\theta)=1 \qquad (-1 \text{ makes } \theta \text{ negative}).$$

$$\therefore\ 90^\circ-\theta=45^\circ. \quad \therefore\ \theta=45^\circ. \quad \therefore\ 2\theta=90^\circ.$$

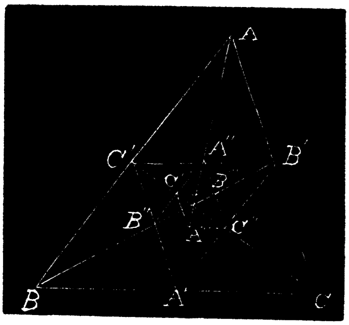
This problem was also solved in an excellent manner by *G. B. M. ZERR.*

64. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates we have $\beta-\gamma=0$, $\gamma-\alpha-0$, $\alpha-\beta=0$ for the equations to AA', BB', CC' respectively.



$$\left(0,\ \frac{2\triangle}{b+c},\ \frac{2\triangle}{b+c}\right),\ \left(\frac{2\triangle}{a+c},\ 0,\ \frac{2\triangle}{a+c}\right),$$

$$\left(\frac{2\triangle}{a+b},\ \frac{2\triangle}{a+b},\ 0\right)$$

are the coordinates of A', B', C' respectively.

$\therefore\ \alpha+\beta-\gamma=0$, $\alpha+\gamma-\beta=0$, $\beta+\gamma-\alpha=0$ are the equations to $A'B'$, $A'C'$, $B'C'$ respectively.

$$\left(\frac{4\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c},\ \frac{2\triangle}{2a+b+c}\right),\ \left(\frac{2\triangle}{a+2b+c},\ \frac{4\triangle}{a+2b+c},\ \frac{2\triangle}{a+2b+c}\right),$$

$$\left(\frac{2\triangle}{a+b+2c},\ \frac{2\triangle}{a+b+2c},\ \frac{4\triangle}{a+b+2c}\right)$$

are the coordinates of A'', B'', C'' respectively.

$\therefore\ \alpha+\beta-3\gamma=0$, $\alpha+\gamma-3\beta=0$, $\beta+\gamma-3\alpha=0$ are the equations to $A''C''$, $A''C''$, $B''C''$ respectively.

$$\left(\frac{4\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c},\ \frac{6\triangle}{2a+3b+3c}\right),\ \left(\frac{6\triangle}{3a+2b+3c},\ \frac{4\triangle}{3a+2b+3c},\right.$$

$$\frac{6\triangle}{3a+2b+3c}\Big),\ \Big(\frac{6\triangle}{3a+3b+2c},\ \frac{6\triangle}{3a+3b+2c},\ \frac{4\triangle}{3a+3b+2c}\Big)$$

are the coordinates of A''', B''', C''' respectively.

$\therefore$ $3\alpha+3\beta-5\gamma=0$, $3\alpha+3\gamma-5\beta=0$, $3\beta+3\gamma-5\alpha=0$ are the equations to $A'''B'''$, $A'''C'''$, $B'''C'''$ respectively.

$$\Big(\frac{12\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c},\ \frac{10\triangle}{6a+5b+5c}\Big),\ \Big(\frac{10\triangle}{5a+6b+5c},\ \frac{12\triangle}{5a+6b+5c},$$

$$\frac{10\triangle}{5a+6b+5c}\Big),\ \Big(\frac{10\triangle}{5a+5b+6c},\ \frac{10\triangle}{5a+5b+6c},\ \frac{12\triangle}{5a+5b+6c}\Big)$$

are the coordinates of A'''', B'''', C'''' respectively.

$\therefore$ $5\alpha+5\beta-11\gamma=0$, $5\alpha+5\gamma-11\beta=0$, $5\beta+5\gamma-11\alpha=0$ are the equations to $A''''B''''$, $A''''C''''$, $B''''C''''$ respectively.

In what follows, the upper signs are used for m odd, and the lower for m even. The mth term of the series 1, 3, 5, 11, 21, 43, 85, etc., is $\frac{1}{3}(2^m\pm1)$.

$$\therefore\ \Big(\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},\quad \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)},$$

$$\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)a+(2^m\pm1)(b+c)}\Big),\quad \Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)},\quad \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)b+(2^m\pm1)(a+c)}\Big),$$

$$\Big(\frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},\quad \frac{2\triangle(2^m\pm1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)},$$

$$\frac{4\triangle(2^{m-1}\mp1)}{2(2^{m-1}\mp1)c+(2^m\pm1)(a+b)}\Big)$$

are the coordinates of A^m, B^m, C^m respectively.

$\therefore$ $\frac{1}{3}(2^m\pm1)(\alpha+\beta)-\frac{1}{3}(2^{m+1}\mp1)\gamma=0$, $\frac{1}{3}(2^m\pm1)(\alpha+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\beta=0$, $\frac{1}{3}(2^m\pm1)(\beta+\gamma)-\frac{1}{3}(2^{m+1}\mp1)\alpha=0$, (1, 2, 3) are the equations to A^mB^m, A^mC^m, B^mC^m respectively.

From (1) and (2), (1) and (3), (2) and (3) respectively, we get

$$\tan A^m=\frac{3\{2^{m+1}(2^{m-1}\mp1)\sin A+2^m(2^m\pm1)(\sin B+\sin C)\}}{3(2^{2m}-1)+2(5.2^{2m-1}\mp2^m+1)\cos A-2(2^m\pm1)(2^{m-1}\mp1)(\cos B+\cos C)}$$

$$\tan B^m = \frac{3\{2^{m+1}(2^{m-1}\mp 1)\sin B + 2^m(2^m \pm 1)(\sin A + \sin C)\}}{3(2^{2m}-1) + 2(5.2^{2m-1}\mp 2^m + 1)\cos B - 2(2^m \pm 1)(2^{m-1}\mp 1)(\cos A + \cos C)}$$

$$\tan C^m = \frac{3\{2^{m+1}(2^{m-1}\mp 1)\sin C + 2^m(2^m \pm 1)(\sin A + \sin B)\}}{3(2^{2m}-1) + 2(5.2^{2m-1}\mp 2^m + 1)\cos C - 2(2^m \pm 1)(2^{m-1}\mp 1)(\cos A + \cos B)}$$

Let A=area of $A^{(m)}B^{(m)}C^{(m)}$, p=perpendicular from $C^{(m)}$ on $A^{(m)}B^{(m)}$.

$$\therefore\ A = [27abc\triangle.2^m] \div [\{2(2^{m-1}\mp 1)a + (2^m \pm 1)(b+c)\}$$
$$\{2(2^{m-1}\mp 1)b + (2^m \pm 1)(a+c)\}\{2(2^{m-1}\mp 1)c + (2^m \pm 1)(a+b)\}]$$

$$p = [9\triangle.2^{m+1}] \div [\{(2^m \pm 1)(a+b) + 2(2^{m-1}\mp 1)c\}$$
$$\sqrt{3(2^{2m+1}+1) + 2(2^m \pm 1)(2^{m+1}\mp 1)(\cos A + \cos B) - 2(2^m \pm 1)^2 \cos C}].$$

But $A = \frac{1}{2}pA^{(m)}B^{(m)}$. $\therefore\ A^{(m)}B^{(m)} = 2A/p$.

$$\therefore\ A^{(m)}B^{(m)}$$
$$= \frac{3abc\sqrt{3(2^{2m+1}+1) + 2(2^m \pm 1)(2^{m+1}\mp 1)(\cos A + \cos B) - 2(2^m \pm 1)^2 \cos C}}{\{2(2^{m-1}\mp 1)a + (2^m \pm 1)(b+c)\}\{2(2^{m-1}\mp 1)b + (2^m \pm 1)(a+c)\}}$$

$$A^{(m)}C^{(m)} = \frac{3abc\sqrt{3(2^{2m+1}+1) + 2(2^m \pm 1)(2^{m+1}\mp 1)(\cos A + \cos C) - 2(2^m \pm 1)^2 \cos B}}{\{2(2^{m-1}\mp 1)a + (2^m \pm 1)(b+c)\}\{2(2^{m-1}\mp 1)c + (2^m \pm 1)(a+b)\}}$$

$$B^{(m)}C^{(m)} = \frac{3abc\sqrt{3(2^{2m+1}+1) + 2(2^m \pm 1)(2^{m+1}\mp 1)(\cos B + \cos B) - 2(2^m \pm 1)^2 \cos A}}{\{2(2^{m-1}\mp 1)b + (2^m \pm 1)(a+c)\}\{2(2^{m-1}\mp 1)c + (2^m \pm 1)(a+b)\}}$$

All principles necessary to understand the above solution will be found in the chapter on "Trilinear Coordinates" in Todhunter's Conic Sections.